

# HIGHER RANK BN-THEORY FOR CURVES OF GENUS 5

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ABSTRACT. In this paper, we consider higher rank Brill-Noether theory for smooth curves of genus 5, obtaining new upper bounds for non-emptiness of Brill-Noether loci and many new examples.

## 1. INTRODUCTION

Let  $C$  be a smooth complex projective curve and let  $B(n, d, k)$  denote the *Brill-Noether locus* of stable bundles on  $C$  of rank  $n$  and degree  $d$  with at least  $k$  independent sections (for the formal definition, see Section 2). This locus has a natural structure as a subscheme of the moduli space of stable bundles on  $C$  of rank  $n$  and degree  $d$ .

In the case  $n = 1$ , the Brill-Noether loci are classical objects. For  $n > 1$ , the study began towards the end of the 1980s and the situation is much less clear, even on a general curve, and there is a great deal that is not known. The problem is completely solved only for  $g \leq 3$  (see [4, 10, 12] and Proposition 2.1), although there are strong results for hyperelliptic and bielliptic curves (see [5] and [1]) and for  $g = 4$  (see [9]).

Our object in this paper is to extend the results of [9] to non-hyperelliptic curves of genus 5. The main results of the paper concern new upper bounds on  $k$  for the non-emptiness of  $B(n, d, k)$  and the corresponding loci  $\tilde{B}(n, d, k)$  for semistable bundles. Since a complete answer is known for  $d \leq 2n$  (see Proposition 2.1), it is sufficient in view of Serre duality to restrict to the range  $2n < d \leq 4n$ . To state our results, it is necessary to distinguish the case of trigonal curves from that of curves of Clifford index 2. For trigonal curves, we have

**Theorem 4.10.** *Let  $C$  be a trigonal curve of genus 5. If  $2n < d \leq 4n$  and  $\tilde{B}(n, d, k) \neq \emptyset$ , then one of the following holds.*

- (i)  $2n < d \leq \frac{7n}{3}$  and  $k \leq n + \frac{1}{4}(d - n)$ ;
- (ii)  $\frac{7n}{3} < d \leq \frac{5n}{2}$  and  $k \leq d - n$ ;
- (iii)  $\frac{5n}{2} \leq d \leq \frac{8n}{3}$  and  $k \leq \frac{3n}{2}$ ;

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- (iv)  $\frac{8n}{3} \leq d < 3n$  and  $k \leq \frac{n}{2} + \frac{3d}{8}$ ;
- (v)  $d = 3n$  and  $k \leq 2n$ ;
- (vi)  $3n < d < 4n$  and  $k \leq \frac{2}{5}(2n + d)$ ;
- (vii)  $d = 4n$  and  $k \leq 2n$ .

If  $B(n, d, k) \neq \emptyset$ , (v) can be replaced by

- (v)'  $d = 3n$  and either  $k \leq \frac{8n}{5}$  or  $(n, d, k) = (1, 3, 2)$ .

These new upper bounds look complicated and can be best appreciated from Figures 1 and 2 in Section 7. They are probably not best possible, but they represent a substantial improvement on the known bound  $k \leq \frac{1}{2}(d + n)$  (see Proposition 2.4), especially in the range  $2n \leq d \leq 3n$ . Note that there is no reason why the optimal upper bound should take a simple form.

For curves of Clifford index 2, we have a somewhat simpler result.

**Theorem 5.2.** *Let  $C$  be a curve of genus 5 and Clifford index 2. If  $2n < d \leq 4n$  and  $\tilde{B}(n, d, k) \neq \emptyset$ , then one of the following holds.*

- (i)  $2n < d \leq \frac{7n}{3}$  and  $k \leq n + \frac{1}{4}(d - n)$ ;
- (ii)  $\frac{7n}{3} < d \leq \frac{5n}{2}$  and  $k < d - n$ ;
- (iii)  $\frac{5n}{2} < d \leq 4n$  and  $k \leq n + \frac{1}{3}(d - n)$ .

For a general curve, this theorem can be slightly improved in the range  $\frac{5n}{2} < d < 3n$  by replacing (iii) by parts (iii) and (iv) from Theorem 4.10 (see Theorem 5.4). In any case, Theorem 5.2 provides an improvement on the known bound  $k \leq n + \frac{1}{3}(d - n)$  (see Proposition 2.7) in the range  $2n < d < \frac{5n}{2}$ . For a graphical representation, see Figures 3 and 4 in Section 7. Again the results are almost certainly not best possible.

We also produce a large number of examples of stable bundles which come close to attaining the upper bounds of Theorems 4.10 and 5.2. Many of these are constructed using elementary transformations, the only problem here being to prove stability. Some of these were already established in [9], but others are new.

In Section 2, we give some background and describe some known results. In Section 3, we obtain upper bounds and also some existence results for non-hyperelliptic curves of genus 5 in general. Section 4 contains results for trigonal curves of genus 5, which are especially strong in the range  $2n < d \leq 3n$ . For curves of genus 5 and Clifford index 2 (see Section 5), the results are quite similar for  $2n < d \leq 3n$ , but are considerably stronger for  $3n < d \leq 4n$ . In Section 6, we consider bundles which maximise the number of sections for given rank and degree, bundles of ranks 2 and 3 and bundles of rank  $n$  with  $h^0 = n + 1$ . Finally, in Section 7, we provide a graphical representation of our results.

Our methods are inspired in particular by those of [5] and work of Mercat [10, 12].

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## 2. BACKGROUND AND SOME KNOWN RESULTS

Let  $C$  be a smooth projective curve of genus  $g$ . Denote by  $M(n, d)$  the moduli space of stable vector bundles of rank  $n$  and degree  $d$  and by  $\widetilde{M}(n, d)$  the moduli space of S-equivalence classes of semistable bundles of rank  $n$  and degree  $d$ . For any integer  $k \geq 1$  we define

$$B(n, d, k) := \{E \in M(n, d) \mid h^0(E) \geq k\}$$

and

$$\widetilde{B}(n, d, k) := \{[E] \in \widetilde{M}(n, d) \mid h^0(\text{gr}E) \geq k\},$$

where  $[E]$  denotes the S-equivalence class of  $E$  and  $\text{gr}E$  is the graded object defined by a Jordan-Hölder filtration of  $E$ . The locus  $B(n, d, k)$  has an *expected dimension*

$$\beta(n, d, k) := n^2(g - 1) + 1 - k(k - d + n(g - 1)),$$

known as the *Brill-Noether number*. For any vector bundle  $E$  on  $C$ , we write  $n_E$  for the rank of  $E$ ,  $d_E$  for the degree of  $E$  and  $\mu(E) = \frac{d_E}{n_E}$  for the slope of  $E$ . The vector bundle  $E$  is said to be *generated* if the evaluation map  $H^0(E) \otimes \mathcal{O}_C \rightarrow E$  is surjective.

We recall the *dual span construction* (see, for example, [6] and [10]), defined as follows. Let  $L$  be a generated line bundle on  $C$  with  $h^0(L) \geq 2$ . Consider the evaluation sequence

$$(2.1) \quad 0 \rightarrow E_L^* \rightarrow H^0(L) \otimes \mathcal{O}_C \rightarrow L \rightarrow 0.$$

Then  $E_L$  is a bundle of rank  $h^0(L) - 1$  and degree  $d_L$  with  $h^0(E) \geq h^0(L)$ . It is called the *dual span of  $L$*  and is also denoted by  $D(L)$ . Although  $E_L$  is not necessarily stable, this is frequently the case.

We begin by recalling some known results. In investigating the non-emptiness of  $B(n, d, k)$  and  $\widetilde{B}(n, d, k)$ , it is sufficient by Serre duality and Riemann-Roch to consider the case  $d \leq n(g - 1)$ . For  $g = 0$  and  $g = 1$ , there is nothing to be done. For  $g = 2$  and  $g = 3$ , a complete solution is known (see [4, 10, 12]). For  $g = 4$ , some strong results were obtained in [9]. For future reference, we note some facts here.

**Proposition 2.1.** *Let  $C$  be a curve of genus  $g \geq 3$  and suppose  $k \geq 1$ .*

- (i) *If  $0 < d < 2n$ , then  $\widetilde{B}(n, d, k) \neq \emptyset$  if and only if  $k - n \leq \frac{1}{g}(d - n)$ . Moreover  $B(n, d, k) \neq \emptyset$  under the same conditions except when  $(n, d, k) = (n, n, n)$  with  $n \geq 2$ .*
- (ii) *If  $C$  is non-hyperelliptic and  $d = 2n$ , then  $\widetilde{B}(n, d, k) \neq \emptyset$  if and only if  $k \leq \frac{ng}{g-1}$ .*
- (iii) *If  $C$  is non-hyperelliptic and  $d = 2n$ , then  $B(n, d, k) \neq \emptyset$  if and only if  $k \leq \frac{n(g+1)}{g}$  or  $(n, d, k) = (g - 1, 2g - 2, g)$ . Moreover  $B(g - 1, 2g - 2, g) = \{D(K_C)\}$ .*

This is contained in [4, 10, 12] and is also included in [9, Propositions 2.1 and 2.2].

**Corollary 2.2.** *If  $2n < d \leq 3n$  and  $k - n \leq \frac{1}{g}(d - 2n)$ , then  $B(n, d, k) \neq \emptyset$ .*

*Proof.*  $B(n, d', k) \neq \emptyset$  for  $n < d' \leq 2n$  and  $k - n \leq \frac{1}{g}(d' - n)$  by Proposition 2.1(i) and (iii). Tensoring by an effective line bundle of degree 1 gives the result.  $\square$

For hyperelliptic curves, a complete solution for  $0 \leq d < 4n$  is contained in [9, Propositions 2.1, 2.2 and 2.3]. This fully covers the cases  $g \leq 4$  and can be completed for  $g = 5$  by the following proposition.

**Proposition 2.3.** *Let  $C$  be a hyperelliptic curve of genus  $g \geq 5$  and  $d = 4n$ . Then  $B(n, d, k) \neq \emptyset$  if and only if either  $k \leq 2n$  or  $(n, d, k) = (1, 4, 3)$ . Moreover,  $\tilde{B}(n, d, k) \neq \emptyset$  if and only if  $k \leq 3n$ .*

*Proof.* The necessity of the condition for  $B(n, d, k)$  is a special case of [5, Theorem 6.2(2)]. The semistable case is easily deducible from this. For sufficiency, take  $s = 2$  in [5, Theorem 6.1].  $\square$

We turn now to non-hyperelliptic curves.

**Proposition 2.4.** *Let  $E$  be a semistable bundle on a non-hyperelliptic curve  $C$  of rank  $n$  and degree  $d$ .*

- (i) *If  $1 \leq \mu(E) \leq 2g - 3$ , then  $h^0(E) \leq \frac{1}{2}(d + n)$ .*
- (ii) *If  $\mu(E) \geq 3$ , then  $h^0(E) \leq d - n$ .*

*Proof.* See [9, Proposition 3.1 and Lemma 3.2]. (Part (i) is contained in [14].)  $\square$

**Lemma 2.5.** *Suppose that  $N$  is a generated line bundle on  $C$  with  $h^0(N) = 2$ . Then, for any bundle  $E$ ,*

$$h^0(N \otimes E) \geq 2h^0(E) - h^0(N^* \otimes E).$$

*In particular, if  $E$  is either semistable with  $\mu(E) < d_N$  or stable of rank  $> 1$  with  $\mu(E) \leq d_N$ , then*

$$h^0(N \otimes E) \geq 2h^0(E).$$

*Proof.* We have an exact sequence

$$0 \rightarrow N^* \otimes E \rightarrow H^0(N) \otimes E \rightarrow N \otimes E \rightarrow 0.$$

The first assertion follows immediately from this. The second assertion follows, if we note that under the stated conditions  $h^0(N^* \otimes E) = 0$ .  $\square$

**Proposition 2.6.** *Let  $C$  be a trigonal curve of genus  $g$  and  $3n < d < 5n$ . If  $k \leq 2 \left\lfloor n + \frac{1}{g}(d - 4n) \right\rfloor$  and  $(n, d, k) \neq (n, 4n, 2n)$  or  $(n, 4n, 2n - 1)$ , then  $B(n, d, k) \neq \emptyset$ .*

*Proof.* We know by Proposition 2.1 that  $B(n, d', k') \neq \emptyset$  if  $0 < d' < 2n$  and  $k' \leq n + \frac{1}{g}(d' - n)$ , except when  $(n, d', k') = (n, n, n)$  with  $n \geq 2$ . Now take  $N$  in Lemma 2.5 to be a trigonal bundle. The result follows from this and the fact that  $B(n, d, k+1) \subset B(n, d, k)$ .  $\square$

For curves of higher Clifford index we have a stronger version of Proposition 2.4(i).

**Proposition 2.7.** *Suppose that  $\text{Cliff}(C) \geq 2$  and  $E$  is a semistable bundle on  $C$  of rank  $n$  and slope  $\mu = \frac{d}{n}$ .*

(i) *If  $2 + \frac{2}{g-4} \leq \mu \leq 2g - 4 - \frac{2}{g-4}$ , then*

$$h^0(E) \leq \frac{d}{2}.$$

(ii) *If  $1 \leq \mu \leq 2 + \frac{2}{g-4}$ , then*

$$h^0(E) \leq \frac{1}{g-2}(d-n) + n.$$

For the proof, see [13, Theorem 2.1]. We have stated this result in full, although only (ii) is relevant for  $g = 5$ .

**Proposition 2.8.** *Let  $C$  be a bielliptic curve and  $n, d$  and  $k$  positive integers.*

- (i) *If  $k \leq \frac{d}{2}$ , then there exists a semistable bundle  $E$  of rank  $n$  and degree  $d$  with  $h^0(E) \geq k$ .*
- (ii) *If  $k < \frac{d}{2}$ , then there exists a stable bundle of rank  $n$  and degree  $d$  with  $h^0(E) \geq k$ .*

This is [13, Theorem 3.1] and is due to Ballico [1, Theorem 5.3 and Proposition 5.4].

A common method of construction is that of elementary transformations. We have in particular

**Proposition 2.9.** *Let  $C$  be a curve of genus  $g \geq 2$  and  $L_1, \dots, L_n$  line bundles of degree  $d$  on  $C$  with  $L_i \not\cong L_j$  for  $i \neq j$  and let  $t > 0$ . Then*

- (i) *there exist stable bundles  $E$  fitting into an exact sequence*

$$0 \rightarrow L_1 \oplus \dots \oplus L_n \rightarrow E \rightarrow \tau \rightarrow 0$$

*where  $\tau$  is a torsion sheaf of length  $t$ ;*

- (ii) *there exist stable bundles  $E$  fitting into an exact sequence*

$$0 \rightarrow E \rightarrow L_1 \oplus \dots \oplus L_n \rightarrow \tau \rightarrow 0$$

*with  $\tau$  as above.*

*Proof.* (i) is a particular case of [11, Théorème A.5]. (ii) can be deduced by replacing each  $L_i$  by  $K_C \otimes L_i$  and using Serre duality. (For a general curve, this is proved in [15].)  $\square$

Finally, we have the following simple lemma.

**Lemma 2.10.** *If  $\tilde{B}(n, d, k) \neq \emptyset$ , then  $B(n', d', k') \neq \emptyset$  for some  $(n', d', k')$  with  $n' \leq n$ ,  $\frac{d'}{n'} = \frac{d}{n}$  and  $\frac{k'}{n'} \geq \frac{k}{n}$ .*

*Proof.* Let  $E$  be a semistable bundle of type  $(n, d, k)$ . At least one factor in any Jordan-Hölder filtration of  $E$  must belong to some  $B(n', d', k')$  as specified in the statement.  $\square$

### 3. NON-HYPERELLIPTIC CURVES OF GENUS 5

In view of the facts cited in section 2, we need to consider only the case  $2 < \mu \leq 4$ .

**Lemma 3.1.** *Let  $C$  be a non-hyperelliptic curve of genus 5 and  $L = K_C(-p)$  for some  $p \in C$ . Then  $L$  is generated,  $E_L$  is stable of rank 3 and degree 7 and*

$$h^0(E_L) = 4.$$

*Proof.*  $L$  is generated since  $C$  is non-hyperelliptic; moreover  $h^0(L) = 4$  by Riemann-Roch. Hence  $E_L$  has rank 3 and degree 7; moreover  $E_L$  is stable by [9, Lemma 3.7]. The fact that  $h^0(E_L) \geq 4$  follows from dualizing (2.1). If  $\text{Cliff}(C) = 2$ , then  $h^0(E_L) \leq 4$  by Proposition 2.7 (ii). Suppose therefore that  $C$  is trigonal with trigonal bundle  $T$ . Since  $h^0(T) = 2$ , Serre duality and Riemann-Roch give  $h^0(K_C \otimes T^*) = 3$ . Hence  $h^0(L \otimes T^*) \geq 2$  and there exist non-zero homomorphisms  $T \rightarrow L$ . Thus we obtain a non-zero homomorphism  $D(L) \rightarrow D(T)$ , i.e.  $E_L \rightarrow T$ . Since  $E_L$  is stable, this must be surjective and we have an exact sequence

$$0 \rightarrow F \rightarrow E_L \rightarrow T \rightarrow 0.$$

The rank-2 bundle  $F$  is semistable, since a line subbundle of  $F$  of degree  $\geq 3$  would contradict the stability of  $E_L$ . But now  $h^0(F) \leq \frac{5}{2}$  by Proposition 2.1(ii), which implies that  $h^0(E_L) \leq 4$ .  $\square$

The following is the case  $g = 5$  of [9, Lemma 3.7(2)].

**Lemma 3.2.** *Let  $C$  be a non-hyperelliptic curve of genus 5 and  $L = K_C(-p)$  for some  $p \in C$ . Suppose that  $E$  is a bundle of rank  $n$  and degree  $d$  with  $h^1(E \otimes L) = 0$  and  $h^0(E) > n + \frac{1}{4}(d - n)$ . Then  $h^0(E_L^* \otimes E) > 0$ .*

The following proposition incorporates the case  $g = 5$  of [9, Lemma 3.8].

**Proposition 3.3.** *Let  $C$  be a non-hyperelliptic curve of genus 5 and  $L = K_C(-p)$  for some  $p \in C$ . Let  $E$  be a semistable bundle of rank  $n$  and degree  $d$  with slope  $\mu > 2$ . Suppose that*

$$h^0(E) > n + \frac{1}{4}(d - n).$$

*Then*

- (i)  $h^0(E_L^* \otimes E) > 0$ ;

- (ii)  $\mu > \frac{7}{3}$ ;
- (iii) if  $\mu < \frac{5}{2}$ ,  $E_L$  can be embedded as a subbundle in  $E$ .

*Proof.* Since  $E \otimes L$  is semistable of slope  $> 2g - 1$ , we have  $h^1(E \otimes L) = 0$ . The assertion (i) now follows from Lemma 3.2. The inequality  $\mu \geq \frac{7}{3}$  and (iii) follow from Lemma 3.1. When  $\mu = \frac{7}{3}$ , (iii) implies that  $E_L$  can be embedded as a subbundle of  $E$ . Hence  $E/E_L$  satisfies the hypotheses of the proposition and so by induction every factor of the Jordan-Hölder filtration of  $E$  is isomorphic to  $E_L$ . Since  $h^0(E_L) = 4$  by Lemma 3.1, this contradicts the hypothesis.  $\square$

**Proposition 3.4.** *Let  $C$  be a non-hyperelliptic curve of genus 5. Suppose that  $k = n + \frac{1}{4}(d - n)$ . Suppose further that  $2 < \frac{d}{n} \leq \frac{7}{3}$ . If  $B(n, d, k) \neq \emptyset$ , then  $(n, d, k) = (3, 7, 4)$ . Moreover,*

$$(3.1) \quad B(3, 7, 4) = \{E_L \mid L = K_C(-p) \text{ for some } p \in C\}.$$

*Proof.* (This follows the same lines as [9, Proposition 6.1], but is more complicated, so we give the proof in full.) Suppose  $E \in B(n, d, k)$ . Note that we have  $h^0(E) = k$  by Proposition 3.3. We first claim that  $E$  is generated.

If not, there exists an exact sequence

$$0 \rightarrow F \rightarrow E \rightarrow \mathbb{C}_q \rightarrow 0$$

with  $h^0(F) = k$ . Let  $L = K_C(-p)$ . Since  $E \otimes L$  is stable with slope  $> 9$ , it follows that  $E \otimes L$  is generated. Hence

$$h^1(F \otimes L) = h^1(E \otimes L) = 0.$$

It now follows from Lemma 3.2 that  $h^0(E_L^* \otimes F) > 0$ . Hence  $E \simeq E_L$ . This contradicts the assumption that  $E$  is not generated.

It follows that we have an exact sequence

$$0 \rightarrow G^* \rightarrow H^0(E) \otimes \mathcal{O}_C \rightarrow E \rightarrow 0$$

with  $n_G = k - n$ ,  $d_G = d$  and  $h^0(G) \geq k$ . It follows that  $K_C \otimes G^*$  has rank  $k - n$  and

$$\begin{aligned} h^0(K_C \otimes G^*) &= h^0(G) - d + 4(k - n) \\ &\geq k - d + d - n = n_G. \end{aligned}$$

Any such bundle necessarily has a section with a zero. So  $K_C \otimes G^*$  admits a line subbundle  $M$  with  $h^0(M) \geq 1$  and  $d_M \geq 1$  and we get

the diagram

$$\begin{array}{ccccccc}
0 & \longrightarrow & E^* & \longrightarrow & W \otimes \mathcal{O}_C & \longrightarrow & G \longrightarrow 0 \\
& & \downarrow \alpha & & \downarrow & & \downarrow \\
0 & \longrightarrow & H^* & \longrightarrow & V \otimes \mathcal{O}_C & \longrightarrow & K_C \otimes M^* \longrightarrow 0 \\
& & & & \downarrow & & \downarrow \\
& & & & 0 & & 0
\end{array}$$

where  $W$  is a subspace of  $H^0(G)$  of dimension  $k$  and  $V$  is the image of  $W$  in  $H^0(K_C \otimes M^*)$ . Now  $K_C \otimes M^*$  is not isomorphic to  $\mathcal{O}_C$ , since  $h^0(G^*) = 0$ . Hence  $\dim V \geq 2$  and  $d_{K_C \otimes M^*} \geq 3$ , since  $C$  is non-hyperelliptic. Since  $d_{K_C \otimes M^*} \leq 7$ , we have also  $\dim V \leq 4$  with equality only if  $K_C \otimes M^* \simeq K_C(-p)$  for some  $p \in C$ .

If  $\alpha = 0$ , then  $E^*$  maps into  $W' \otimes \mathcal{O}_C$ , where  $W = W' \oplus V'$  and  $V'$  maps isomorphically to  $V$ . It follows that  $V' \otimes \mathcal{O}_C$  maps to a trivial direct summand of  $G$  contradicting the fact that  $h^0(G^*) = 0$ . So  $\alpha \neq 0$ .

If  $\dim V = 2$ , then  $\alpha(E^*)$  is a quotient line bundle of  $E^*$  of degree  $\leq -3$ , contradicting the stability of  $E$ .

If  $\dim V = 4$ , we can write  $K_C \otimes M^* \simeq K_C(-p) =: L$  and then  $H \simeq E_L$ , which is stable with  $\mu(E_L) \geq \mu(E)$ . Hence  $E \simeq E_L$ .

It remains to consider the case  $\dim V = 3$ . If  $K_C \otimes M^* \simeq K_C(-p) =: L$  for some  $p \in C$ , then  $H^*$  is a subbundle of  $E_L^*$ , so there exists a non-zero homomorphism  $E^* \rightarrow E_L^*$  implying  $E \simeq E_L$ . Finally suppose that  $\dim V = 3$  and  $V = H^0(K_C \otimes M^*)$ . Then  $d_H = d_{K_C \otimes M^*} = 5$  or  $6$ . If  $\alpha$  has rank 2, this contradicts the stability of  $E$ . So suppose  $\text{rk } \alpha = 1$ . Since  $H$  is generated and  $H^0(H^*) = 0$ , every quotient line bundle of  $H$  has degree  $\geq 3$ , again contradicting the stability of  $E$ .  $\square$

**Proposition 3.5.** *Let  $C$  be a non-hyperelliptic curve of genus 5 and  $E$  a semistable bundle of rank  $n$  and degree  $d$  with slope  $\mu(E) > \frac{7}{3}$ . Then*

$$h^0(E) \leq d - n.$$

*Proof.* The proof is by induction on  $n$ . Note that by Proposition 2.4(ii) we can assume that  $\mu(E) < 3$ .

For  $n = 1$ , the result is trivial. For  $n = 2$ , the only possibility is  $d = 5$  and then  $h^0(E) \leq 3$  by Proposition 2.4(i). For  $n = 3$ , the only possibility is  $d = 8$  and then  $h^0(E) \leq 5$  by Proposition 2.4(i).

Suppose now  $n \geq 4$  and the proposition is proved for rank  $\leq n - 1$ . Then there exists an exact sequence

$$0 \rightarrow F \rightarrow E \rightarrow G \rightarrow 0$$

in which  $F$  is a proper subbundle of maximal slope and is stable. Moreover  $G$  is semistable. To see this, suppose that  $G'$  is a quotient bundle of  $G$  with  $\mu(G') < \mu(G)$ . Set  $F' := \ker(E \rightarrow G')$ . Note that  $n_{F'} > n_F$ . We have

$$\mu(F') \leq \mu(F).$$



Also

$$\mu(F') \leq \mu(G')$$

by semistability of  $E$ . So

$$\begin{aligned} d &= n_F \mu(F) + (n - n_F) \mu(G) \\ &> n_F \mu(F') + (n - n_F) \mu(G') \\ &\geq n_{F'} \mu(F') + (n - n_{F'}) \mu(G') = d, \end{aligned}$$

a contradiction.

If  $h^0(E) > d - n$ , then, by Proposition 3.3, there exists a non-zero homomorphism  $E_L \rightarrow E$ . So

$$\mu(F) \geq \mu(E_L) = \frac{7}{3},$$

with equality only if  $F \simeq E_L$ . Also

$$\mu(G) > \frac{7}{3}$$

by semistability of  $E$ . So  $G$  satisfies the inductive hypotheses and  $F$  does, unless  $F \simeq E_L$  in which case  $h^0(F) = d_F - n_F$ . This completes the inductive step and hence the proof.  $\square$

**Lemma 3.6.** *Let  $C$  be a non-hyperelliptic curve of genus 5. There exists a stable bundle  $U$  on  $C$  of rank 2 with  $d_U = 5$  and  $h^0(U) = 3$  if and only if  $C$  is trigonal. Moreover  $U \simeq D(K_C \otimes T^*)$ , where  $T$  is the unique trigonal bundle.*

*Proof.* Let  $E$  be a stable bundle of rank 2 with  $d_E = 5$  and  $h^0(E) = 3$ . Using Proposition 2.1(ii), it is easy to see that the evaluation map  $H^0(E) \otimes \mathcal{O}_C \rightarrow E$  is surjective. It follows that  $E \simeq E_M$  for some line bundle  $M$  of degree 5 with  $h^0(M) = 3$ . Then  $K_C \otimes M^*$  has degree 3 and  $h^0 = 2$ . In other words, it is the trigonal bundle.

Hence, if there exists a stable rank-2 bundle  $E$  of degree 5 with  $h^0(E) = 3$ , then  $C$  is trigonal and  $E \simeq D(K_C \otimes T^*)$ . Now there exist non-trivial extensions

$$(3.2) \quad 0 \rightarrow K_C \otimes T^{*2} \rightarrow U \rightarrow T \rightarrow 0.$$

For any such extension,  $U$  is stable. Moreover, there exists an extension for which all sections of  $T$  lift to  $U$ , since

$$(3.3) \quad H^0(T) \otimes H^0(T^2) \rightarrow H^0(T^3)$$

is not surjective. To see this, note first that  $H^0(T) \otimes H^0(T^2)$  has dimension 6 and  $h^0(T^3) = 5$ . The kernel of (3.3) is isomorphic to  $H^0(T)$  by the base-point free pencil trick, so has dimension 2.  $\square$

**Lemma 3.7.** *Let  $C$  be a non-hyperelliptic curve of genus 5. Let  $E$  be a semistable bundle on  $C$  of slope  $\mu$ ,  $2 < \mu < 3$  such that  $h^0(E) > n + \frac{1}{4}(d - n)$ . Then one of the following occurs.*

- (i)  $\mu > \frac{7}{3}$  and  $E_L$  can be embedded as a subbundle of  $E$ ;

- (ii)  $\mu \geq \frac{8}{3}$  and  $E$  has a rank-3 subbundle  $E'$  of degree 8;
- (iii)  $C$  is trigonal,  $\mu \geq \frac{5}{2}$  and  $U$  can be embedded as a subbundle of  $E$ .

*Proof.* By Proposition 3.3(i) and (ii),  $h^0(E_L^* \otimes E) > 0$  and  $\mu > \frac{7}{3}$ . Let  $E_L \rightarrow E$  be a non-zero homomorphism. If this homomorphism does not embed  $E_L$  as a subbundle, then either (ii) holds or the image of  $E_L$  in  $E$  is stable of rank 2 and degree 5. By Lemma 3.6, this leaves (iii) as the only possibility.  $\square$

**Proposition 3.8.** *Let  $C$  be a non-hyperelliptic curve of genus 5. Then  $B(n, d, k) \neq \emptyset$  in the following cases.*

- (i)  $(n, d, k) = (4r + s, 8r + 2s + 1, 5r + s)$  for  $1 \leq r \leq 4, s \geq 0$ ;
- (ii)  $(n, d, k) = (4r + s, 8r + 2s + 2, 5r + s)$  for  $1 \leq r \leq 4, s \geq 4r + 1$ ;
- (iii)  $(n, d, k) = (n, 2n + 1, n + r)$  for  $n \geq 5r, r \geq 1$ .

*Proof.* These are special cases of [9, Propositions 3.3 and 3.6 and Example 3.9].  $\square$

**Proposition 3.9.** *Let  $C$  be a curve of genus 5. Then  $B(n, d, k) \neq \emptyset$  for  $3n < d < 4n$  and  $k \leq d - 2n$ .*

*Proof.* Take  $L_1, \dots, L_n$  pairwise non-isomorphic line bundles of degree 4 with  $h^0(L_i) = 2$ . Such bundles exist for all  $n$  on any curve of genus 5. The result follows from Proposition 2.9(ii).  $\square$

The argument used in the proof of Proposition 3.9 does not work for  $d = 4n$ . However, we have the following proposition.

**Proposition 3.10.** *Let  $C$  be a curve of genus 5. Then*

- (i)  $\tilde{B}(n, 4n, k) \neq \emptyset$  for  $k \leq 2n$ ;
- (ii)  $B(n, 4n, k) \neq \emptyset$  for  $k \leq \frac{6n}{5}$  and  $B(4, 16, 5) \neq \emptyset$ ;
- (iii)  $B(n, 4n, k) \neq \emptyset$  for  $k < 2n$  if  $C$  is general.

*Proof.* For (i) we can take a direct sum of line bundles of degree 4 with  $h^0 = 2$ . (ii) follows from Proposition 2.1(iii) by tensoring with an effective line bundle of degree 2. (iii) is proved in [15].  $\square$

#### 4. TRIGONAL CURVES OF GENUS 5

In this section, let  $C$  be a trigonal curve of genus 5 with trigonal bundle  $T$  and let  $U = D(K_C \otimes T^*)$ .

**Lemma 4.1.** *The bundle  $U$  admits a unique line subbundle  $M$  of degree 2. Moreover  $M \simeq K_C \otimes T^{*2}$ .*

*Proof.* By (3.2), it is sufficient to show that  $h^0(U^* \otimes T) = 1$ . However,  $U^* \otimes T$  is stable of rank 2 and degree 1, so  $h^0(U^* \otimes T) \leq 1$  by Proposition 2.1(i). The result follows.  $\square$

**Lemma 4.2.** *Let  $L = K_C(-p)$  for some  $p \in C$ . Then there exist surjective homomorphisms  $E_L \rightarrow T$  and  $E_L \rightarrow U$ . Moreover,  $U$  is the only quotient of  $E_L$  of rank 2 and degree 5.*

*Proof.* The existence of  $E_L \rightarrow T$  was proved in the proof of Lemma 3.1. Using (2.1), we have an exact sequence

$$0 \rightarrow H^0(E_L^* \otimes U) \rightarrow H^0(L) \otimes H^0(U) \rightarrow H^0(L \otimes U).$$

Now  $h^0(L) = 4$ ,  $h^0(U) = 3$  and  $h^0(L \otimes U) = 11$  by Riemann-Roch. So there exists a non-zero homomorphism  $E_L \rightarrow U$ . If this homomorphism has rank 1, then  $E_L$  has a quotient of rank 1 and degree  $\leq 2$ , contradicting stability. If  $E_L \rightarrow U$  has rank 2, but is not surjective, then  $E_L$  has a line subbundle of degree  $\geq 3$ , again contradicting stability.

Now any rank-2 and degree-5 quotient bundle of  $E_L$  must be stable with  $h^0 \geq 3$ . By Lemma 3.6 the only such bundle is  $U$ .  $\square$

**Proposition 4.3.** *Let  $E$  be a semistable bundle with  $\frac{5}{2} \leq \mu(E) < 3$ . Then*

$$h^0(E) \leq \max \left\{ \frac{n}{2} + \frac{3d}{8}, \frac{3n}{2} \right\}.$$

*Proof.* Suppose first that  $\mu(E) \geq \frac{8}{3}$ . In this case  $2 < \mu(K_C \otimes T^* \otimes E^*) \leq \frac{7}{3}$ . Hence by Proposition 3.3,

$$h^0(K_C \otimes T^* \otimes E^*) \leq n + \frac{1}{4}(4n - d) = 2n - \frac{d}{4}.$$

By Lemma 2.5,

$$h^0(T \otimes E) \geq 2h^0(E).$$

Hence by Riemann-Roch,

$$d - n = \chi(T \otimes E) \geq 2h^0(E) - \left(2n - \frac{d}{4}\right),$$

which implies the result in this case.

If  $\frac{5}{2} \leq \mu(E) < \frac{8}{3}$ , we argue in the same way, but now use Proposition 3.5 to show that  $h^0(K_C \otimes T^* \otimes E^*) \leq 4n - d$ . Then

$$d - n = \chi(T \otimes E) \geq 2h^0(E) - 4n + d.$$

Hence  $h^0(E) \leq \frac{3n}{2}$ .  $\square$

**Lemma 4.4.** *Let  $L = K_C(-p)$  for some  $p \in C$ . Then the multiplication map*

$$(4.1) \quad H^0(T) \otimes H^0(K_C \otimes E_L^*) \rightarrow H^0(T \otimes K_C \otimes E_L^*)$$

*is surjective with kernel  $H^0(T^* \otimes K_C \otimes E_L^*)$  of dimension 4.*

*Proof.* From the sequence

$$0 \rightarrow T^* \rightarrow H^0(T) \otimes \mathcal{O}_C \rightarrow T \rightarrow 0,$$

we see that the kernel of (4.1) is  $H^0(T^* \otimes K_C \otimes E_L^*)$ . Since  $T^* \otimes K_C \otimes E_L^*$  is a stable bundle of rank 3 and degree 8, Proposition 4.3 implies that  $h^0(T^* \otimes K_C \otimes E_L^*) \leq 4$ .

Now  $h^0(T) = 2$ ,  $h^0(K_C \otimes E_L^*) = h^0(E_L) - d_{E_L} + 12 = 9$  and  $h^0(T \otimes K_C \otimes E_L^*) = 14$  by Riemann-Roch. The result follows.  $\square$

**Lemma 4.5.** *Let  $L = K_C(-p)$  for some  $p \in C$ . Then the multiplication map*

$$(4.2) \quad H^0(U) \otimes H^0(K_C \otimes E_L^*) \rightarrow H^0(U \otimes K_C \otimes E_L^*)$$

*is surjective with kernel  $h^0(T \otimes E_L^*)$  of dimension 2.*

*Proof.* From the sequence

$$0 \rightarrow (K_C \otimes T^*)^* \rightarrow H^0(U) \otimes \mathcal{O}_C \rightarrow U \rightarrow 0,$$

we see that the kernel of (4.2) is  $H^0(T \otimes E_L^*)$ . Since  $T \otimes E_L^*$  is stable of rank 3 and degree 2, we have  $h^0(T \otimes E_L^*) \leq 2$  by Proposition 2.1(i).

Now  $h^0(U) = 3$ ,  $h^0(K_C \otimes E_L^*) = 9$  and, by Riemann-Roch,  $h^0(U \otimes K_C \otimes E_L^*) = 25$ . The result follows.  $\square$

**Proposition 4.6.** *Suppose that  $E$  is a semistable bundle of rank  $2r$  and degree  $5r$  with  $h^0(E) = 3r$ . Then*

$$E \simeq \bigoplus_{i=1}^r U.$$

*Proof.* First we claim that  $h^0(U^* \otimes E) > 0$ .

Since  $U \simeq D(K_C \otimes T^*)$  by Lemma 3.6, we have an exact sequence

$$0 \rightarrow U^* \otimes E \rightarrow H^0(K_C \otimes T^*) \otimes E \rightarrow K_C \otimes T^* \otimes E \rightarrow 0.$$

Now  $h^0(K_C \otimes T^*) = 3$  and  $h^0(E) = 3r$ . On the other hand,  $T \otimes E^*$  is a semistable bundle of rank  $2r$  and degree  $r$ , so by Proposition 2.1(i),  $h^0(T \otimes E^*) \leq \frac{9r}{5}$ . It follows by Riemann-Roch that  $h^0(K_C \otimes T^* \otimes E) \leq 9r - \frac{r}{5}$ . Hence  $h^0(U^* \otimes E) \geq \frac{r}{5}$ , which proves the claim.

The proof of the proposition is by induction on  $r$ . For  $r = 1$ , we have a non-zero homomorphism  $U \rightarrow E$  which is necessarily an isomorphism.

Suppose therefore that  $r \geq 2$  and the result is proved for the case  $r - 1$ . A non-zero homomorphism  $U \rightarrow E$  is necessarily an injection onto a subbundle. By the inductive hypothesis,  $E/U$  is isomorphic to  $r - 1$  copies of  $U$  and we have an exact sequence

$$0 \rightarrow U \rightarrow E \rightarrow \bigoplus_{i=1}^{r-1} U \rightarrow 0;$$

moreover all sections of  $\bigoplus_{i=1}^{r-1} U$  must lift. If the extension is non-trivial, it follows that the map

$$H^0(U) \otimes H^0(K_C \otimes U^*) \rightarrow H^0(K_C \otimes U \otimes U^*)$$

is not surjective. Its kernel is  $H^0(K_C \otimes U^* \otimes (K_C \otimes T^*)^*) = H^0(T \otimes U^*)$ , which has dimension  $\leq 1$  by Proposition 2.1(i).

Now  $h^0(U) = 3$ ,  $h^0(K_C \otimes U^*) = 6$  and  $h^0(K_C \otimes U \otimes U^*) = 17$  by Riemann-Roch, since  $h^0(U \otimes U^*) = 1$ . This gives a contradiction. It follows that the extension is trivial and  $E \simeq \bigoplus_{i=1}^r U$ .  $\square$

**Lemma 4.7.** *Let  $E$  be a stable bundle with  $\mu(E) = 3$  and  $n \geq 2$ . Then*

$$h^0(E) \leq \frac{8n}{5}.$$

*Proof.* Suppose  $E$  is a stable bundle of rank  $n \geq 2$ ,  $\mu(E) = 3$  and  $h^0(E) > \frac{8n}{5}$ . By Lemma 2.5,

$$h^0(T \otimes E) \geq 2h^0(E) > \frac{16n}{5}.$$

By Riemann-Roch and Serre duality,

$$h^0(K_C \otimes T^* \otimes E^*) > \frac{6n}{5}.$$

Since  $K_C \otimes T^* \otimes E^*$  is stable of slope 2, this contradicts Proposition 2.1(iii), unless  $E \simeq K_C \otimes T^* \otimes D(K_C)^*$ . However  $h^0(D(K_C)) = 5$ . It follows that in this case  $h^0(T \otimes E) = 13$ . Hence  $h^0(E) \leq 6 < \frac{8n}{5}$ , since  $n = 4$ , and the result still holds.  $\square$

**Lemma 4.8.** *There exist generated line bundles of degree 4 with  $h^0 = 2$ .*

*Proof.* Let  $Q := K_C \otimes T^*(-p)$  for some  $p \in C$ . Certainly  $h^0(Q) = 2$ . If  $Q$  is not generated, then  $Q = T(q)$  for some  $q \in C$ . So  $K_C = T^2(p+q)$  which is true for a unique divisor  $p+q$ . This implies the assertion.  $\square$

**Proposition 4.9.** *Let  $E$  be a semistable bundle of rank  $n$  and degree  $d$ . Suppose  $3 < \mu(E) \leq 4$ . Then*

$$h^0(E) \leq \begin{cases} \frac{2}{5}(2n+d) & \text{if } \mu(E) < 4 \\ 2n & \text{if } \mu(E) = 4. \end{cases}$$

*Proof.* Let  $Q$  be a generated line bundle with  $d_Q = 4$  and  $h^0 = 2$ . Suppose first that  $\mu(E) < 4$ . Then, since  $0 < \mu(K_C \otimes Q^* \otimes E^*) < 1$ , we have

$$h^0(K_C \otimes Q^* \otimes E^*) \leq n + \frac{1}{5}(3n-d)$$

by Proposition 2.1(i). Therefore, by Riemann-Roch,

$$h^0(Q \otimes E) \leq \frac{4}{5}(2n+d).$$

From Lemma 2.5, we get

$$2h^0(E) \leq h^0(Q \otimes E),$$

which implies the assertion in this case.

Now suppose  $\mu(E) = 4$ . In view of Lemma 2.10, we can suppose moreover that  $E$  is stable. If  $n = 1$ , then obviously  $h^0(E) \leq 2$ . If

$n \geq 2$ , then  $\mu(K_C \otimes Q^* \otimes E^*) = 0$ ; hence  $h^0(K_C \otimes Q^* \otimes E^*) = 0$ . By Serre duality and Riemann-Roch, we obtain  $h^0(Q \otimes E) = 4n$ , giving the assertion.  $\square$

The following theorem summarizes the results on upper bounds obtained above (see Figures 1 and 2 in Section 7).

**Theorem 4.10.** *Let  $C$  be a trigonal curve of genus 5. If  $2n < d \leq 4n$  and  $\tilde{B}(n, d, k) \neq \emptyset$ , then one of the following holds.*

- (i)  $2n < d \leq \frac{7n}{3}$  and  $k \leq n + \frac{1}{4}(d - n)$ ;
- (ii)  $\frac{7n}{3} < d \leq \frac{5n}{2}$  and  $k \leq d - n$ ;
- (iii)  $\frac{5n}{2} \leq d \leq \frac{8n}{3}$  and  $k \leq \frac{3n}{2}$ ;
- (iv)  $\frac{8n}{3} \leq d < 3n$  and  $k \leq \frac{n}{2} + \frac{3d}{8}$ ;
- (v)  $d = 3n$  and  $k \leq 2n$ ;
- (vi)  $3n < d < 4n$  and  $k \leq \frac{2}{5}(2n + d)$ ;
- (vii)  $d = 4n$  and  $k \leq 2n$ .

If  $B(n, d, k) \neq \emptyset$ , (v) can be replaced by

- (v)'  $d = 3n$  and either  $k \leq \frac{8n}{5}$  or  $(n, d, k) = (1, 3, 2)$ .

*Proof.* (i) follows from Proposition 3.3, (ii) is Proposition 3.5, for (iii) and (iv) see Proposition 4.3, for (v) and (v)' combine Lemma 4.7 with the existence of  $T$ , and for (vi) and (vii) see Proposition 4.9.  $\square$

**Remark 4.11.** Let  $\text{Cliff}_n(C)$  be the rank- $n$  Clifford index as defined for example in [7]. It follows from Theorem 4.10 that all bundles computing  $\text{Cliff}_n(C)$  must have degree  $3n$  and  $h^0 = 2n$ . In fact, by [7, Corollary 4.8], there is only one such bundle, namely  $\oplus_{i=1}^n T$ .

**Proposition 4.12.**  $B(2, 7, 4) = \emptyset$ .

*Proof.* Let  $E \in B(2, 7, 4)$ . We prove first that

$$h^0(T^* \otimes E) > 0.$$

In fact,  $H^0(T^* \otimes E)$  is the kernel of the multiplication map

$$H^0(T) \otimes H^0(E) \rightarrow H^0(T \otimes E).$$

Now  $h^0(T) = 2$ ,  $h^0(E) = 4$  and, by Riemann-Roch,

$$h^0(T \otimes E) = h^0(K_C \otimes T^* \otimes E^*) + 7 + 6 - 8.$$

Since  $K_C \otimes T^* \otimes E^*$  is a stable bundle of rank 2 and degree 3, we have

$$h^0(K_C \otimes T^* \otimes E^*) \leq 2$$

by Proposition 2.1(i) and hence  $h^0(T \otimes E) \leq 7$ . This proves the assertion.

It follows that we have an exact sequence

$$(4.3) \quad 0 \rightarrow T \rightarrow E \rightarrow M \rightarrow 0$$

with  $d_M = 4$ ,  $h^0(M) = 2$  and all sections of  $M$  lift. Hence the map

$$H^0(M) \otimes H^0(K_C \otimes T^*) \rightarrow H^0(K_C \otimes M \otimes T^*)$$

is not surjective. However,  $h^0(K_C \otimes T^*) = 3$  and the kernel is  $H^0(K_C \otimes T^* \otimes M^*)$ , which has dimension  $\leq 1$ .

Moreover,  $h^0(K_C \otimes M \otimes T^*) = 5$  by Riemann-Roch, a contradiction. This proves that  $B(2, 7, 4) = \emptyset$ .  $\square$

**Proposition 4.13.**  $B(2, 8, 4) \neq \emptyset$ .

*Proof.* We consider non-trivial extensions (4.3) with  $M = K_C \otimes T^*$ . Then certainly  $E$  is semistable. Hence  $h^0(E) \leq 4$  by Proposition 4.9. Since  $h^0(M) = 3$ , it follows that not all sections of  $M$  lift. So the canonical map

$$(4.4) \quad H^1(M^* \otimes T) \rightarrow \text{Hom}(H^0(M), H^1(T))$$

is injective. Noting that  $M$  is generated, choose a 2-dimensional subspace  $V$  of  $H^0(M)$  which generates  $M$ . We obtain an exact sequence

$$0 \rightarrow H^0(\mathcal{O}_C) \rightarrow V \otimes H^0(M) \rightarrow H^0(M^2).$$

Since  $\dim V = 2$ ,  $h^0(M) = 3$  and  $h^0(M^2) = 6$ , it follows that  $V \otimes H^0(M) \rightarrow H^0(M^2)$  is not surjective and has cokernel of dimension 1. Equivalently, the dual map

$$H^1(M^* \otimes T) \rightarrow \text{Hom}(V, H^1(T))$$

has kernel of dimension 1. Taking (4.3) to be the extension corresponding to a non-zero element  $\xi$  of this kernel, we obtain a unique bundle  $E$  with  $h^0(E) = 4$ .

We need to show that  $E$  is stable. If  $E$  is not stable, then  $E$  must have a tetragonal subbundle  $Q$  admitting a non-zero homomorphism  $Q \rightarrow M$ . This implies that (4.3) becomes trivial, when pulled back by  $Q \rightarrow M$ . It follows that the element of  $H^1(M^* \otimes T)$  defining (4.3) is in the kernel of the map

$$H^1(M^* \otimes T) \rightarrow H^1(Q^* \otimes T) \rightarrow \text{Hom}(H^0(Q), H^1(T)).$$

$H^0(Q)$  and  $V$  are both subspaces of codimension 1 of  $H^0(M)$  and  $\xi$  goes to zero under the restriction of (4.4) to both  $H^0(Q)$  and  $V$ . Hence  $H^0(Q) = V$ , in which case  $V$  does not generate  $M$ . This contradicts the assumption. The conclusion is that  $B(2, 8, 4) \neq \emptyset$ .  $\square$

**Proposition 4.14.**

$$B(n, 3n, k) \neq \emptyset \quad \text{for } k \leq 2 \left\lfloor \frac{6n}{5} \right\rfloor - n.$$

Moreover,  $B(4, 12, 6) \neq \emptyset$ .

*Proof.* We know, by Proposition 2.1(iii), that  $B(n, 2n, k') \neq \emptyset$  for  $k' \leq \frac{6n}{5}$ . If  $E \in B(n, 2n, k')$ , then  $h^0(T \otimes E) \geq 2h^0(E)$  by Lemma 2.5 and so

$$h^0(K_C \otimes T^* \otimes E^*) \geq 2h^0(E) - n.$$

This proves the first statement. For the second, note that  $K_C \otimes T^* \otimes D(K_C)^*$  belongs to  $B(4, 12, 6)$ .  $\square$

**Proposition 4.15.**  $B(n, d, k) \neq \emptyset$  in the following cases.

- (i)  $(n, d, k) = (4r + s, 12r + 3s - 1, 6r + s - 1)$  for  $1 \leq r \leq 4$ ,  $s \geq 0$ ;
- (ii)  $(n, d, k) = (4r + s, 12r + 3s - 2, 6r + s - 2)$  for  $1 \leq r \leq 4$ ,  $s \geq 4r + 1$ ;
- (iii)  $(n, d, k) = (n, 3n - 1, n + 2r - 1)$  for  $n \geq 5r$ ,  $r \geq 1$ .

*Proof.* By Proposition 3.8(i), we have with the hypotheses of (i),  $B(4r + s, 8r + 2s + 1, 5r + s) \neq \emptyset$ . Using Lemma 2.5 with  $N = T$ , it follows that  $B(4r + s, 20r + 5s + 1, 10r + 2s) \neq \emptyset$ . The result follows by Serre duality and Riemann-Roch. (ii) and (iii) follow similarly from Proposition 3.8(ii) and (iii).  $\square$

**Proposition 4.16.** For any  $p \in C$ , there exist exact sequences

$$(4.5) \quad 0 \rightarrow U \rightarrow E \rightarrow \mathbb{C}_p \rightarrow 0$$

with  $E$  stable. Hence  $B(2, 6, 3) \neq \emptyset$ .

*Proof.* Consider exact sequences (4.5). If  $E$  is not stable, then it possesses a line subbundle  $N$  of degree 3. By Lemma 4.1, it follows that we have a diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & U & \longrightarrow & E & \longrightarrow & \mathbb{C}_p \longrightarrow 0 \\ & & \uparrow & & \uparrow & & \parallel \\ 0 & \longrightarrow & K_C \otimes T^{*2} & \longrightarrow & N & \longrightarrow & \mathbb{C}_p \longrightarrow 0 \end{array}$$

and the embedding of  $K_C \otimes T^{*2}$  in  $U$  is unique up to a scalar. It follows that such a diagram cannot exist for the general extension (4.5).  $\square$

**Proposition 4.17.**  $B(2r, 6r - 1, 3r - 1) \neq \emptyset$  for any  $r \geq 1$ .

*Proof.* Choose  $r$  pairwise non-isomorphic bundles  $E_1, \dots, E_r \in B(2, 6, 3)$ . These exist by Proposition 4.16. Let  $E$  be an elementary transformation

$$0 \rightarrow E \rightarrow E_1 \oplus \dots \oplus E_r \rightarrow \mathbb{C}_p \rightarrow 0$$

for some  $p \in C$  such that the homomorphisms  $E_i \rightarrow \mathbb{C}$  are all non-zero. Since the partial direct sums of the  $E_i$  are the only subbundles of  $E_1 \oplus \dots \oplus E_r$  of slope 3, it follows that every subbundle of  $E$  has slope  $< 3$ . Hence  $E \in B(2r, 6r - 1, 3r - 1)$ .  $\square$

**Proposition 4.18.** For any  $p \in C$ , there exist exact sequences

$$(4.6) \quad 0 \rightarrow U \oplus U \rightarrow E \rightarrow \mathbb{C}_p \rightarrow 0$$

with  $E$  stable. Hence  $B(4, 11, 6) \neq \emptyset$ .

*Proof.* Consider exact sequences (4.6). If  $E$  is not stable, there exists a diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & U \oplus U & \longrightarrow & E & \longrightarrow & \mathbb{C}_p \longrightarrow 0 \\ & & \uparrow & & \uparrow & & \parallel \\ 0 & \longrightarrow & F' & \longrightarrow & F & \longrightarrow & \mathbb{C}_p \longrightarrow 0 \end{array}$$



with  $n_F \leq 3$  and  $\mu(F) = 3$ .

If  $n_F = 1$ , then  $d_{F'} = 2$  and  $h^0(F'^* \otimes (U \oplus U)) = 2$  by Lemma 4.1. If  $n_F = 2$ , then  $d_{F'} = 5$ . In this case,  $F' \simeq U$  and again  $h^0(F'^* \otimes (U \oplus U)) = 2$ . In both cases the diagram cannot exist for a general extension (4.6). If  $n_F = 3$ , then  $d_{F'} = 8$ . This contradicts the semistability of  $U \oplus U$ .  $\square$

**Proposition 4.19.** *For any  $p, q \in C$ , there exist exact sequences*

$$(4.7) \quad 0 \rightarrow E_L \rightarrow E \rightarrow \mathbb{C}_q \rightarrow 0$$

*with  $E$  stable, where  $L = K_C(-p)$ . In particular  $B(3, 8, 4) \neq \emptyset$ .*

*Proof.* Suppose that  $F$  is a proper subbundle of  $E$  with  $\mu(F) \geq \frac{8}{3}$ . Then we have a diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & E_L & \longrightarrow & E & \longrightarrow & \mathbb{C}_q \longrightarrow 0 \\ & & \uparrow & & \uparrow & & \parallel \\ 0 & \longrightarrow & F' & \longrightarrow & F & \longrightarrow & \mathbb{C}_q \longrightarrow 0 \end{array}$$

If  $n_F = 2$ , we must have  $d_{F'} = 5$ . This contradicts the stability of  $E_L$ .

If  $n_F = 1$ , we must have  $d_{F'} = 2$ . It follows from Lemma 4.2 that  $F' \simeq T(-p)$ . Moreover,  $h^0(E_L \otimes F'^*) \leq 2$  by Proposition 2.1(i), since  $E_L \otimes F'^*$  is stable of slope  $\frac{1}{3}$ . It follows that the diagram cannot exist for a general extension (4.7).  $\square$

## 5. CURVES OF CLIFFORD INDEX 2

Suppose that  $C$  is a curve of genus 5 and Clifford index 2. The main difference from the trigonal case is that the bundles  $T$  and  $U$  do not exist. However  $C$  is tetragonal and possesses a one-dimensional family of line bundles of degree 4 with  $h^0 = 2$ . In this case we have the following slight improvement of Proposition 3.5.

**Proposition 5.1.** *Let  $C$  be a curve of genus 5 and Clifford index 2 and  $E$  a semistable bundle of rank  $n$  and degree  $d$  with  $\mu(E) > \frac{7}{3}$ . Then*

$$h^0(E) < d - n.$$

*Proof.* The proof of Proposition 3.5 goes through with the improved inequality, noting that, by Proposition 2.7(ii) and Lemma 3.6 there are no bundles  $E$  of ranks 1, 2 or 3 with  $\mu(E) > \frac{7}{3}$  and  $h^0(E) = d - n$ .  $\square$

**Theorem 5.2.** *Let  $C$  be a curve of genus 5 and Clifford index 2. If  $2n < d \leq 4n$  and  $\tilde{B}(n, d, k) \neq \emptyset$ , then one of the following holds.*

- (i)  $2n < d \leq \frac{7n}{3}$  and  $k \leq n + \frac{1}{4}(d - n)$ ;
- (ii)  $\frac{7n}{3} < d \leq \frac{5n}{2}$  and  $k < d - n$ ;
- (iii)  $\frac{5n}{2} < d \leq 4n$  and  $k \leq n + \frac{1}{3}(d - n)$ .

*Proof.* (i) follows from Proposition 3.3, for (ii) see Proposition 5.1 and (iii) is Proposition 2.7(ii) for  $g = 5$ .  $\square$

**Remark 5.3.** It follows from Theorem 5.2 that all bundles computing  $\text{Cliff}_n(C)$  in this case have degree  $4n$  and  $h^0 = 2n$ . For rank 2, this was proved in [8, Proposition 5.7]. At least on a general curve of genus 5, there exist stable bundles of rank 2 and degree 8 with  $h^0 = 4$  by [2, Section 3]. This answers a question raised in [8, Remark 5.8]).

**Theorem 5.4.** *Let  $C$  be a general curve of genus 5. If  $2n < d \leq 4n$  and  $\tilde{B}(n, d, k) \neq \emptyset$ , then one of the following holds.*

- (i)  $2n < d \leq \frac{7n}{3}$  and  $k \leq n + \frac{1}{4}(d - n)$ ;
- (ii)  $\frac{7n}{3} < d \leq \frac{5n}{2}$  and  $k < d - n$ ;
- (iii)  $\frac{5n}{2} < d \leq \frac{8n}{3}$  and  $k \leq \frac{3n}{2}$ ;
- (iv)  $\frac{8n}{3} < d < 3n$  and  $k \leq \frac{n}{2} + \frac{3d}{8}$ ;
- (v)  $3n \leq d \leq 4n$  and  $k \leq n + \frac{1}{3}(d - n)$ .

*Proof.* (i), (ii) and (v) follow from Theorem 5.2. (iii) and (iv) follow from Theorem 4.10(iii) and (iv) by semicontinuity.  $\square$

**Proposition 5.5.** *Let  $C$  be a curve of genus 5 and Clifford index 2. Then*

$$(5.1) \quad B(2, 6, 3) \neq \emptyset.$$

*Proof.* Let  $M$  be a line bundle of degree 6 with  $h^0(M) = 3$ . Then  $M$  is generated and the bundle  $E_M$ , defined as in (2.1), has rank 2 and degree 6. If  $E_M$  is not stable, then we have a diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & E_M^* & \longrightarrow & H^0(M) \otimes \mathcal{O}_C & \longrightarrow & M \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \parallel \\ 0 & \longrightarrow & N & \longrightarrow & F & \longrightarrow & M \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \\ & & 0 & & 0 & & \end{array}$$

with  $N$  a line bundle with  $d_N \leq -3$ . It follows that  $F$  is a generated rank-2 bundle with  $d_F \leq 3$  and  $h^0(F) \geq 3$ . This cannot exist on a non-trigonal curve, so  $E_M \in B(2, 6, 3)$ .  $\square$

**Corollary 5.6.**  $B(2r, 6r - 1, 3r - 1) \neq \emptyset$  for any  $r \geq 1$ .

*Proof.* The proof is exactly the same as for Proposition 4.17, using Proposition 5.5 and the fact that  $\beta(2, 6, 3) > 0$ .  $\square$

**Corollary 5.7.**  $B(2r, 6r + 1, 3r) \neq \emptyset$  for any  $r \geq 1$ .

*Proof.* We consider extensions

$$0 \rightarrow E_1 \oplus \cdots \oplus E_r \rightarrow E \rightarrow \mathbb{C}_p \rightarrow 0,$$

where  $E_1, \dots, E_r \in B(2, 6, 3)$  and are pairwise non-isomorphic. The general extension gives a stable bundle.  $\square$

**Lemma 5.8.** *Let  $C$  be a curve of genus 5 and Clifford index 2. Let  $L_i = K_C(-p_i)$  for  $1 \leq i \leq r$ , where  $p_1, \dots, p_r$  are distinct points of  $C$ . Then every proper subbundle  $F$  of  $E_{L_1} \oplus \dots \oplus E_{L_r}$ , which is not isomorphic to a partial direct sum of factors of  $E_{L_1} \oplus \dots \oplus E_{L_r}$ , has*

$$d_F \leq \frac{7}{3}n_F - 1.$$

*Proof.* The proof is by induction on  $r$ . If  $r = 1$ , then  $n_F = 1$  or  $2$ .

If  $n_F = 1$ , then  $d_F \leq 1$ , since otherwise  $E_{L_1}/F$  is a quotient of  $E_{L_1}$  of rank 2 and degree 5 with  $h^0 \geq 3$ . It is easy to see that this quotient must be stable, which contradicts Lemma 3.6. If  $n_F = 2$ , then  $d_F \leq 3$ , since otherwise  $E_{L_1}$  would have a quotient bundle of rank 1 and degree  $\leq 3$  with  $h^0 \geq 2$ . This contradicts  $\text{Cliff}(C) = 2$ .

Now suppose  $r \geq 2$  and the lemma is proved for  $r - 1$  factors. Consider the projection  $\pi : F \rightarrow E_{L_1}$ . We can assume without loss of generality that  $\pi \neq 0$ . If  $\text{rk } \pi = 3$ , then by induction

$$d_F \leq 7 + \frac{7}{3}(n_F - 3) - 1 = \frac{7}{3}n_F - 1.$$

If  $\text{rk } \pi = 1$ , then

$$d_F \leq 1 + \frac{7}{3}(n_F - 1) < \frac{7}{3}n_F - 1.$$

If  $\text{rk } \pi = 2$ , then

$$d_F \leq 3 + \frac{7}{3}(n_F - 2) < \frac{7}{3}n_F - 1.$$

This completes the proof.  $\square$

**Proposition 5.9.** *Let  $C$  be a curve of genus 5 and Clifford index 2. Suppose  $r \geq 1$ ,  $p \in C$  and  $L_1, \dots, L_r$  are as in Lemma 5.8. Let*

$$0 \rightarrow E_{L_1} \oplus \dots \oplus E_{L_r} \rightarrow E \rightarrow \mathbb{C}_p \rightarrow 0$$

*be the extension classified by  $(e_1, \dots, e_r)$ , where the  $e_i \in \text{Ext}(\mathbb{C}_p, E_{L_i})$  are all non-zero. Then  $E$  is stable. Hence*

$$B(3r, 7r + 1, 4r) \neq \emptyset.$$

*Proof.* It follows from Lemma 5.8 that any proper subbundle  $F$  of  $E$  has  $d_F \leq \frac{7}{3}n_F$ . Hence  $E$  is stable.  $\square$

**Proposition 5.10.** *Let  $C$  be a curve of genus 5 and Clifford index 2. Then  $B(3, 9, 4) \neq \emptyset$  and  $B(3, 12, 4) \neq \emptyset$ .*

*Proof.* Let  $L = K_C(-p)$  for some  $p \in C$ . Consider an exact sequence

$$0 \rightarrow E \rightarrow E_L(q) \rightarrow \mathbb{C}_q \rightarrow 0$$

for any  $q \in C$ . Since  $E_L$  as a subsheaf of  $E_L(q)$  consists of local sections vanishing at  $q$ , we have a sheaf inclusion  $E_L \hookrightarrow E$ . This implies that  $h^0(E) \geq 4$ .

Now by Lemma 5.8, any subbundle of  $E_L(q)$  of rank 1 has degree  $\leq 2$  and any subbundle of rank 2 has degree  $\leq 5$ . These do not contradict the stability of  $E$ . So  $E \in B(3, 9, 4)$ . Moreover,  $E(r) \in B(3, 12, 4)$  for any  $r \in C$ .  $\square$

**Remark 5.11.** The stable bundles constructed in this section exist on any curve of genus 5 and Clifford index 2, in particular on a bielliptic curve of genus 5. Many of them have  $k > \frac{d}{2}$ , so lie outside the scope of Ballico's result (Proposition 2.8). This means that the bundles and their sections are not lifted from the corresponding elliptic curve.

## 6. EXTREMAL BUNDLES, BUNDLES OF LOW RANK AND $k = n + 1$

Let  $C$  be a non-hyperelliptic curve of genus 5. The bundles  $D(K_C)$ ,  $E_L$  and  $Q$  are extremal in the sense that they take the maximal value of  $\frac{h^0}{n}$  for bundles of the same slope (see the figures in Section 7). By Proposition 2.1(iii),  $D(K_C)$  is the only stable bundle representing the point  $(2, \frac{5}{4})$  in the BN-map. By Proposition 3.4, the only bundles on the line in the BN-map joining  $(2, \frac{5}{4})$  to  $(\frac{7}{3}, \frac{4}{3})$  are  $D(K_C)$  and the bundles  $E_L$ .

When  $C$  is trigonal, then  $U$  and  $T$  are also extremal. By Remark 4.11,  $T$  is the only stable bundle representing the point  $(3, 2)$ .

**Proposition 6.1.** *Let  $C$  be a non-hyperelliptic curve of genus 5 and  $E$  a semistable bundle of rank  $n$  and degree  $d$  with slope  $\mu(E) > \frac{7}{3}$ . Then*

$$h^0(E) < d - n,$$

*unless  $C$  is trigonal and  $E \simeq \oplus_i U$  or  $E \simeq \oplus_i T$ .*

*Proof.* For  $\text{Cliff}(C) = 2$ , this is Proposition 5.1. For  $C$  trigonal, the proof follows that of Proposition 3.5. For  $n = 1, 2, 3$  the result is clear. Continuing with the proof, we see that either  $h^0(E) < d - n$  or there is an exact sequence

$$0 \rightarrow F \rightarrow E \rightarrow G \rightarrow 0$$

with  $F \simeq E_L, U$  or  $T$  and  $G \simeq \oplus_i T$  or  $\simeq \oplus_i U$  and all sections of  $G$  must lift.

If  $F \simeq T$  and  $G \simeq \oplus_i T$ , then  $E \simeq \oplus_i T$  by Remark 4.11. If  $F \simeq U$  and  $G \simeq \oplus_i T$ , then  $\frac{5}{2} < \mu(E) < 3$  and the result follows from Proposition 4.3. If  $F \simeq U$  and  $G \simeq \oplus_i U$ , then  $E \simeq \oplus_i U$  by Proposition 4.6.

If  $F \simeq E_L$  and  $G \simeq \oplus_i T$ , then by Lemma 4.4 not all sections of  $G$  lift. Finally, if  $F \simeq E_L$  and  $G \simeq \oplus_i U$ , then by Lemma 4.5 not all sections of  $G$  lift. This completes the inductive step and hence the proof.  $\square$

**Proposition 6.2.** *Suppose  $0 < d < 16$ . Then  $B(2, d, k) \neq \emptyset$  if and only if*

$$\beta(2, d, k) := 17 - k(k - d + 8) \geq 0$$

*with the following exceptions.*

- (i)  $(d, k) = (2, 2)$  or  $(14, 8)$ , in which case  $\tilde{B}(2, d, k) \neq \emptyset$  but  $B(2, d, k) = \emptyset$ ;
- (ii)  $B(2, 8, 4) \neq \emptyset$  for  $C$  general and for  $C$  trigonal, but possibly not for all  $C$  with  $\text{Cliff}(C) = 2$ ;  $\tilde{B}(2, 8, 4) \neq \emptyset$  for all  $C$ ;
- (iii) if  $C$  is trigonal,  $B(2, 5, 3) \neq \emptyset$  and  $B(2, 11, 6) \neq \emptyset$ .

*Proof.* Suppose first that  $0 < d \leq 8$ . For  $k = 1$  and  $2$ , the result follows from Proposition 2.1. For  $k = 3$ ,  $\beta(2, d, k) \geq 0$  is equivalent to  $d \geq 6$ . The result follows from Lemma 3.6 and Propositions 4.16 and 5.5. For  $k = 4$ ,  $\beta(2, d, k) \geq 0$  is equivalent to  $d \geq 8$  and the result follows from Propositions 4.12 and 4.13, Theorem 5.2(iii) and Remark 5.3. For  $k \geq 5$ ,  $\beta(2, d, k) < 0$  for all  $d \leq 8$  and all  $\tilde{B}(2, d, k) = \emptyset$  by Theorems 4.10 and 5.2.

For  $d > 8$  we use Serre duality and Riemann-Roch.  $\square$

**Proposition 6.3.** *Suppose  $0 < d < 24$ . Then  $B(3, d, k) \neq \emptyset$  if and only if*

$$\beta(3, d, k) := 37 - k(k - d + 12) \geq 0$$

*with the following exceptions.*

- (i)  $(d, k) = (3, 3)$  or  $(21, 12)$ , in which case  $\tilde{B}(3, d, k) \neq \emptyset$  but  $B(3, d, k) = \emptyset$ ;
- (ii) if  $C$  is trigonal,  $\tilde{B}(3, 9, 4) \neq \emptyset$  and  $\tilde{B}(3, 15, 7) \neq \emptyset$ , but it is possible that  $B(3, 9, 4) = \emptyset$  and  $B(3, 15, 7) = \emptyset$ ;
- (iii) if  $\text{Cliff}(C) = 2$ , it is possible that  $B(3, 9, 5) \neq \emptyset$  and  $B(3, 15, 8) \neq \emptyset$ ;
- (iv)  $B(3, 10, 5)$  and  $B(3, 14, 7)$  might be empty;
- (v)  $\tilde{B}(3, 12, 5) \neq \emptyset$ ; moreover,  $B(3, 12, 5) \neq \emptyset$  for  $C$  general, but there might be curves for which  $B(3, 12, 5) = \emptyset$ ;
- (vi)  $\tilde{B}(3, 12, 6) \neq \emptyset$ , but  $B(3, 12, 6)$  might be empty;
- (vii) if  $C$  is trigonal,  $B(3, 10, 6)$ ,  $B(3, 11, 6)$ ,  $B(3, 13, 7)$  and  $B(3, 14, 8)$  might be non-empty.

*Proof.* Suppose first that  $0 < d \leq 12$ . For  $k = 1, 2, 3$  the result follows from Proposition 2.1.

For  $k = 4$ ,  $\beta(3, d, k) \geq 0$  is equivalent to  $d \geq 7$ . We have  $B(3, d, 4) = \emptyset$  for  $d \leq 6$  by Proposition 2.1,  $B(3, 7, 4) \neq \emptyset$  by Lemma 3.1 and  $B(3, 8, 4) \neq \emptyset$  by Propositions 4.19 and 5.9. If  $\text{Cliff}(C) = 2$ , then  $B(3, 9, 4) \neq \emptyset$  and  $B(3, 12, 4) \neq \emptyset$  by Proposition 5.10. For  $C$  trigonal, we see that  $\tilde{B}(3, 9, 4) \neq \emptyset$  by taking the direct sum of a bundle in  $B(2, 6, 3)$  and a line bundle of degree 3 and  $h^0 = 1$ . It is possible that  $B(3, 9, 4) = \emptyset$ , but  $B(3, 12, 4) \neq \emptyset$  by Proposition 2.6. For  $d \geq 10$ ,  $d \neq 12$ ,  $B(3, d, 4) \neq \emptyset$  by Proposition 3.9 for any  $C$ .

For  $k = 5$ ,  $\beta(3, d, k) \geq 0$  is equivalent to  $d \geq 10$ .  $\tilde{B}(3, d, 5) = \emptyset$  for  $d \leq 8$  by Theorem 4.10(i)-(iv) and Theorem 5.2.  $B(3, 9, 5) = \emptyset$  if  $C$  is trigonal by Theorem 4.10(v)'. We do not know whether this is

true for any curve of Clifford index 2.  $B(3, 11, 5) \neq \emptyset$  in all cases by Proposition 3.9. For  $d = 12$ , use Proposition 3.10.

For  $k = 6$ ,  $\beta(3, d, k) \geq 0$  is equivalent to  $d \geq 12$ . If  $\text{Cliff}(C) = 2$ ,  $\tilde{B}(3, d, 6) = \emptyset$  for  $d \leq 11$  by Theorem 5.2 and for  $d \leq 8$  if  $C$  is trigonal by Theorem 4.10.  $\tilde{B}(3, 12, 6) \neq \emptyset$  in all cases by Proposition 3.10. If  $C$  is trigonal, then  $B(3, 9, 6) = \emptyset$  by Theorem 4.10(v)'.

For  $k \geq 7$ ,  $\beta(3, d, k) < 0$  for all  $d \leq 12$  and  $\tilde{B}(3, d, k) = \emptyset$  by Theorems 4.10 and 5.2.

For  $d > 12$  we use Serre duality and Riemann-Roch.  $\square$

It would be possible to extend this analysis to  $k = 4$ , but the details would be complicated. However, there is one case where there is a simple answer.

**Proposition 6.4.** *Let  $C$  be a non-hyperelliptic curve of genus 5. Then  $\tilde{B}(4, 10, 5) \neq \emptyset$ . If  $\text{Cliff}(C) = 2$ , then  $B(4, 10, 5) \neq \emptyset$  and  $\tilde{B}(4, 10, 5) = B(4, 10, 5)$ .*

*Proof.* Since  $\beta(4, 10, 5) = 10 > 0$ ,  $\tilde{B}(4, 10, 5) \neq \emptyset$  for a general curve by [3, Theorem 5.1]. This holds for any curve by semicontinuity. If  $B(4, 10, 5) = \emptyset$ , then by Lemma 2.10,  $B(2, 5, 3) \neq \emptyset$ . By Lemma 3.6, this is not possible for  $\text{Cliff}(C) = 2$ . Indeed, in this case, there are no strictly semistable bundles of rank 4 and degree 10 with  $h^0 \geq 5$ .  $\square$

This result can be completed and partially extended to the case  $k = n + 1$  for all  $n$ .

**Proposition 6.5.** *Let  $C$  be a non-hyperelliptic curve of genus 5 and suppose  $n \geq 2$ .*

- (i) *If  $\beta(n, d, n + 1) < 0$ , then  $\tilde{B}(n, d, n + 1) = \emptyset$ , except when  $C$  is trigonal and  $(n, d, n + 1) = (2, 5, 3)$ .*
- (ii) *If  $\beta(n, d, n + 1) \geq 0$ , then*
  - (a)  $\tilde{B}(n, d, n + 1) \neq \emptyset$ ,
  - (b)  $B(n, d, n + 1) \neq \emptyset$ , *except possibly when*
    - $n \geq 10, n \text{ even}, d = 2n + 4$ ,
    - $n \geq 9, n \text{ divisible by } 3, d = 2n + 3$ ,
    - $n = 8, d = 18 \text{ or } 20$ ,
    - $n = 6, 14 \leq d \leq 16$ ,
    - $n = 4, d = 10, C \text{ trigonal}$ ,
    - $n = 3, d = 9, C \text{ trigonal}$ .

*Proof.* Suppose first  $n \geq 5$ . Then  $\beta(n, d, n + 1) < 0$  is equivalent to  $d \leq n + 4$ . (i) now follows from Proposition 2.1 and (ii)(a) follows from [3, Theorem 5.1] for  $C$  general and hence for any  $C$ .

For (ii)(b),  $B(n, d, n + 1) \neq \emptyset$  for  $n + 5 \leq d \leq 2n$  by Proposition 2.1. Tensoring by an effective line bundle gives the same result for  $2n + 5 \leq d \leq 3n$  and for  $d = 4n$ . For  $3n + 1 \leq d < 4n$ , see Proposition

3.9. For  $d = 2n + 1$ , see Proposition 3.8(iii) and for  $d = 2n + 2$  and  $n \geq 9$ , Proposition 3.8(ii). For  $d > 4n$ , tensor by an effective line bundle. Note also that, if  $(n, d) = 1$ , then (ii)(b) follows from (ii)(a).

For  $n = 4$ , we have  $\beta(4, d, 5) = 5d - 40$ . If  $d \leq 7$ , then  $\tilde{B}(4, d, 5) = \emptyset$  by Proposition 2.1(i). For  $d = 8$ , note that  $D(K_C) \in B(4, 8, 5)$ . For  $d = 9, 10, 11$ , see Propositions 3.8(i), 6.4, 4.17 and Corollary 5.6. For  $d \geq 12$ , tensor by an effective line bundle. For  $d = 14$ , when  $C$  is trigonal, we still have  $B(4, 14, 5) \neq \emptyset$  by Proposition 2.6.

For  $n = 3$ , see Proposition 6.3 and for  $n = 2$ , see Proposition 6.2.  $\square$

## 7. BN-MAP FOR GENUS 5

The following figures are the most significant part of the BN-map for non-hyperelliptic curves of genus 5. The map plots  $\lambda = \frac{k}{n}$  against  $\mu = \frac{d}{n}$ .

We begin with the trigonal case (Figures 1 and 2).

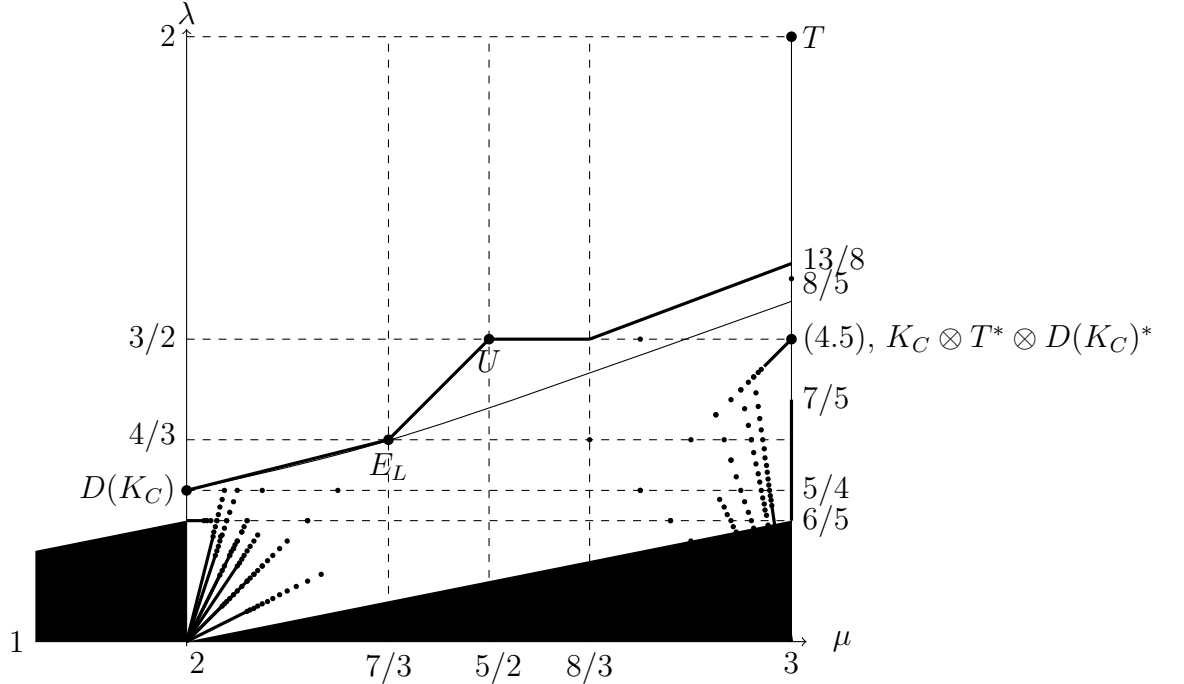
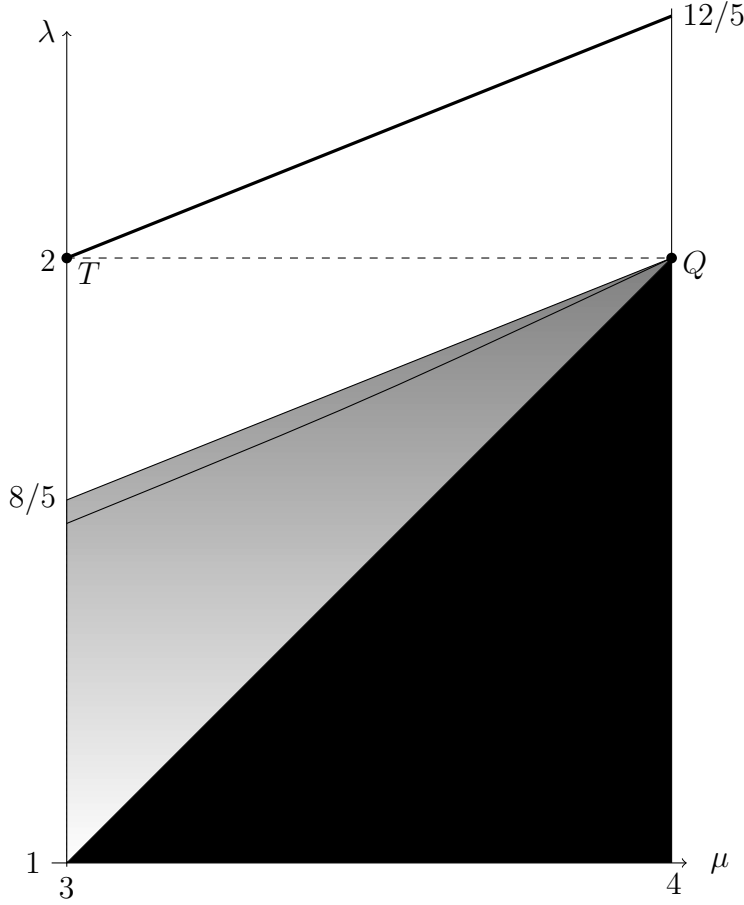


Figure 1:  $C$  trigonal,  $2 \leq \mu \leq 3$

Figure 2:  $C$  trigonal,  $3 < \mu \leq 4$ 

The thicker solid lines indicate the upper bounds for non-emptiness, given by Theorem 4.10.

The shaded areas consist of points  $(\mu, \lambda)$  for which there exist  $(n, d, k)$  with

$$\frac{d}{n} = \mu, \quad \frac{k}{n} = \lambda \quad \text{and} \quad B(n, d, k) \neq \emptyset.$$

The black areas are given by Proposition 2.1, Corollary 2.2 and Proposition 3.9 and all  $B(n, d, k)$  corresponding to points in these areas are non-empty. Note that the vertical line at  $\mu = 4$  in Figure 2 is not included. The vertical line at  $\mu = 3$  in Figure 1 ending at  $\lambda = \frac{7}{5}$  corresponds to Proposition 4.14, but not all  $B(n, 3n, k)$  corresponding to points on this line are non-empty.

In the grey area, which corresponds to Proposition 2.6, there are some  $(n, d, k)$  for which possibly  $B(n, d, k) = \emptyset$ . However, for any  $(\mu, \lambda)$  in this area, there exist  $(n, d, k)$  with  $\mu = \frac{d}{n}$ ,  $\lambda = \frac{k}{n}$  such that  $B(n, d, k) \neq \emptyset$ .

The dots represent points for which some  $B(n, d, k) \neq \emptyset$ . The series of dots arise from Propositions 3.8, 4.15, 4.17. There are also isolated dots corresponding to Propositions 4.18, 4.19 and 6.4. The dot at



$(3, \frac{8}{5})$  may not represent a bundle, but is the upper bound established in Lemma 4.7.

The BN-curve (the thin curve in the figures) given by  $\lambda(\lambda - \mu + 4) = 4$  (or  $\beta(n, d, k) = 1$ ) passes through the points  $(2, -1 + \sqrt{5})$ ,  $(\frac{7}{3}, \frac{4}{3})$ ,  $(\frac{8}{3}, \frac{1}{3}(-2 + \sqrt{40}))$ ,  $(3, \frac{1}{2}(-1 + \sqrt{17}))$  and  $(4, 2)$ . The bundle  $D(K_C)$  in Figures 1 and 3 lies marginally above the curve and corresponds to the value  $\beta = 0$ ; the bundles  $U$  and  $T$  in Figure 1 correspond to the value  $\beta = -1$ . All the bundles constructed in this paper in the case  $\text{Cliff}(C) = 2$  have  $\beta(n, d, k) \geq 0$ , but this does not rule out the possibility that  $B(n, d, k)$  could be non-empty for some  $(n, d, k)$  with  $\beta(n, d, k) < 0$  even in this case.

We turn now to the case of Clifford index 2, represented by Figures 3 and 4. Note that in Figure 3, the thick upper line applies to any curve of Clifford index 2 (Theorem 5.2) and the lower line to a general curve (Theorem 5.4). The vertical line at  $\mu = 4$  in Figure 4 is not included. The series of dots arise from Corollaries 5.6, 5.7 and Proposition 5.9.

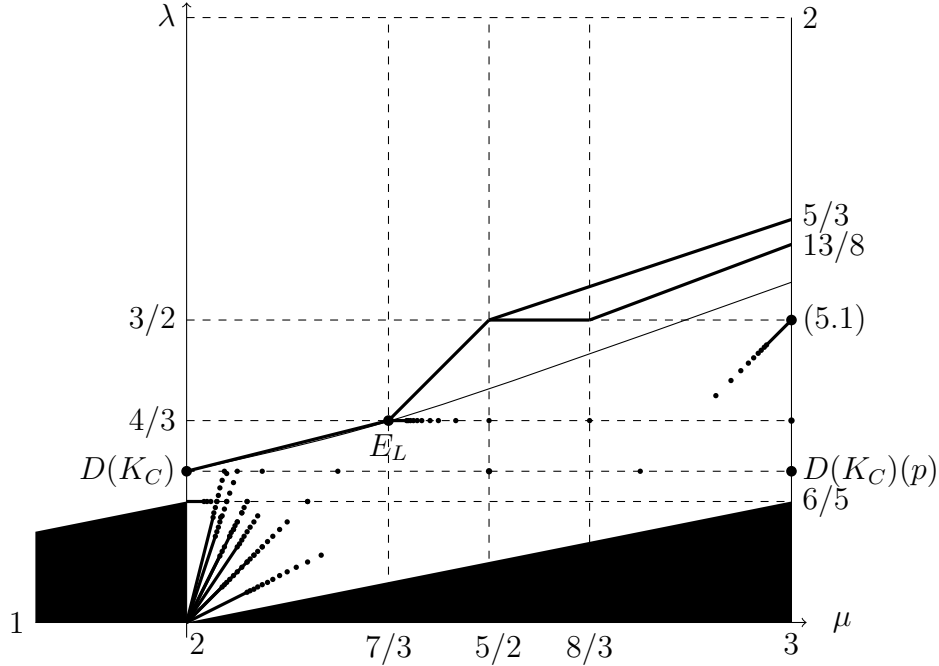


Figure 3:  $\text{Cliff}(C) = 2$ ,  $2 \leq \mu \leq 3$

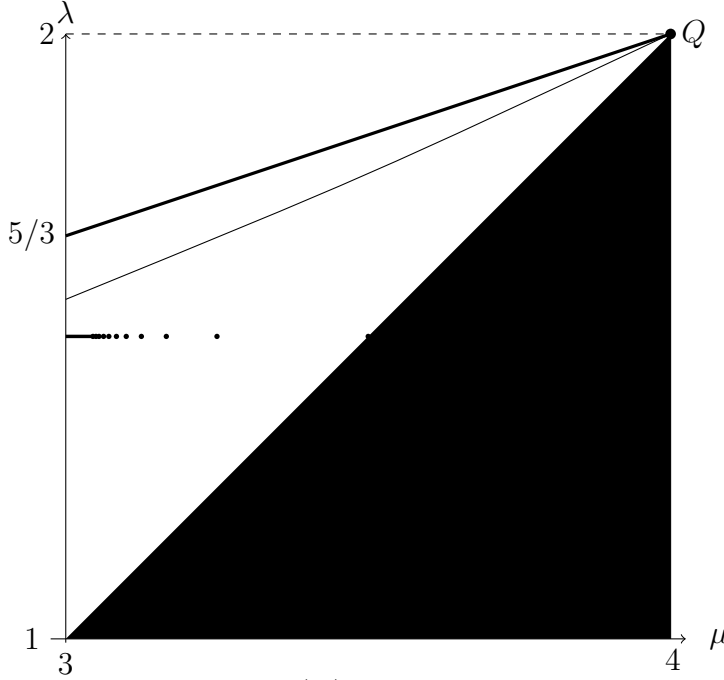


Figure 4:  $\text{Cliff}(C) = 2$ ,  $3 < \mu \leq 4$

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